REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden to Washington Headquarters Service, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188) Washington, DC 20503.
PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ADDRESS.

1. REPORT DATE *(DD-MM-YYYY)*	2. REPORT TYPE	3. DATES COVERED *(From - To)*
MARCH 2010	Conference Paper Postprint	January 2008 – June 2009

4. TITLE AND SUBTITLE

NEW BOUNDS ON THE TOTAL-SQUARED-CORRELATION OF QUATERNARY SIGNATURE SETS AND OPTIMAL DESIGNS

5a. CONTRACT NUMBER
In-House

5b. GRANT NUMBER
FA8750-08-1-0063

5c. PROGRAM ELEMENT NUMBER
62702F

6. AUTHOR(S)

Ming Li, Stella Batalama, Dimitris Pados, and John Matyjas

5d. PROJECT NUMBER
ANCL

5e. TASK NUMBER
62

5f. WORK UNIT NUMBER
07

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

AFRL/RIGF
525 Brooks Road
Rome, NY 13441-4505

State University of New York at Buffalo
Department of Electrical Engineering
332 Bonner Hall
Buffalo, NY 14260-2050

8. PERFORMING ORGANIZATION REPORT NUMBER

N/A

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

AFRL/RIGF
525 Brooks Road
Rome NY 13441-4505

10. SPONSOR/MONITOR'S ACRONYM(S)
N/A

11. SPONSORING/MONITORING AGENCY REPORT NUMBER
AFRL-RI-RS-TP-2010-14

12. DISTRIBUTION AVAILABILITY STATEMENT
APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED. PA #: 88ABW-2008-1044 Date Cleared: November 18, 2008

13. SUPPLEMENTARY NOTES

14. ABSTRACT
We derive new bounds on the total squared correlation (TSC) of quaternary (quadriphase) signature/sequence sets for all lengths and set sizes. Then we design minimum-TSC optimal sets that meet the new bounds with equality. Direct numerical comparison with the TSC value of the recently obtained optimal binary sets shows what gains are materialized by moving from the binary to the quaternary code-division multiplexing alphabet. On the other hand, comparison with the Welch TSC value for real/complex-field sets shows that, arguably, not much is to be gained by raising the alphabet size above four.

15. SUBJECT TERMS
Code-division multiplexing, quaternary alphabet, sequences, Welch bound

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			Michael J. Medley
U	U	U	UU	6	19b. TELEPHONE NUMBER *(Include area code)* N/A

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std. Z39.18

New Bounds on the Total-Squared-Correlation of Quaternary Signature Sets and Optimal Designs

Ming Li, Stella N. Batalama[†], Dimitris A. Pados
Department of Electrical Engineering
State University of New York at Buffalo
Buffalo, NY 14260
E-mail: {mingli,batalama,pados}@eng.buffalo.edu

John D. Matyjas
Air Force Research Laboratory/RIGE
525 Brooks Rd
Rome, NY 13441
E-mail: John.Matyjas@rl.af.mil

Abstract—We derive new bounds on the total squared correlation (TSC) of quaternary (quadriphase) signature/sequence sets for all lengths L and set sizes K. Then, for all K, L, we design minimum-TSC optimal sets that meet the new bounds with equality. Direct numerical comparison with the TSC value of the recently obtained optimal binary sets shows under what K, L realizations gains are materialized by moving from the binary to the quaternary code-division multiplexing alphabet. On the other hand, comparison with the Welch TSC value for real/complex-field sets shows that, arguably, not much is to be gained by raising the alphabet size above four for any K, L.

Index Terms—Code-division multiplexing, quaternary alphabet, sequences, Welch bound.

I. INTRODUCTION

We consider the problem of designing sets of code sequences (signatures) for code-division multiplexing applications from the quaternary alphabet $\{\pm 1, \pm j\}$, $j \triangleq \sqrt{-1}$. A fundamental measure of the cross-correlation properties of a signature set (and subsequently overall code-division system performance) is the total squared correlation (TSC). If $\mathcal{S} \triangleq \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K\}$, $\mathbf{s}_k \in \mathbb{C}^L$, $\|\mathbf{s}_k\| = 1$, $k = 1, 2, \ldots, K$, is a set of K normalized (complex, in general) signatures of length (processing gain) L, then the TSC of $\mathcal{S}$ is the sum of the squared magnitudes of all inner products between signatures,

$$\text{TSC}(\mathcal{S}) \triangleq \sum_{m=1}^{K} \sum_{n=1}^{K} \left| \mathbf{s}_m^H \mathbf{s}_n \right|^2$$

where "H" denotes the conjugate transpose operation. For real/complex-valued signature sets ($\mathcal{S} \in \mathbb{C}^{L \times K}$ or $\mathcal{S} \in \mathbb{R}^{L \times K}$), TSC is bounded from below by [1]-[3]

$$\text{TSC}(\mathcal{S}) \geq \frac{KM}{L} \tag{1}$$

where $M = \max\{K, L\}$. The bound in (1) is called the "Welch bound" and the signature sets that satisfy (1) with equality are called Welch-bound-equality (WBE) sets. While for real/complex-valued signature sets the Welch bound is always achievable [4]-[9], this is not the case in general for finite-alphabet signatures. Recently, new bounds were derived for the TSC of binary (alphabet $\{\pm 1\}$) signature sets for all lengths L and set sizes K together with optimal set designs for (almost) all K and L [10]-[12]. The sum capacity, total asymptotic efficiency, and maximum squared correlation of minimum-TSC

optimal sets were found in [13]-[14]. Minimum-TSC and other digital sequence sets are studied and utilized in [15]-[16].

The gap in TSC between minimum-TSC binary signature sets and Welch-bound-equality real/complex sets can be reduced if we utilize alphabets of larger size. Certainly there is a tradeoff between system performance and transceiver complexity that is associated with our selection of the alphabet size. The quaternary (or quadriphase or 4-phase) alphabet appears as a good candidate since most code-division multiplexing (CDM) systems already employ quadrature signaling. Thus, utilizing quaternary signature sets would not incur significant additional cost over binary signature sets.

In this paper we consider a quaternary alphabet and investigate under what K, L realizations gains can be materialized by moving from the binary to the quaternary code-division multiplexing alphabet. In particular, we first derive new bounds on the TSC of any quaternary signature matrix $\mathcal{S} = [\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K] \in \frac{1}{\sqrt{L}} \{\pm 1, \pm j\}^{L \times K}$ for all possible K and L values.

II. NEW BOUNDS ON THE TSC OF QUATERNARY SIGNATURE SETS

We recall that the Karystinos-Pados bounds on the TSC of a binary signature set can be given compactly by the following expression [10]-[12]

$$\text{TSC}(\mathcal{S}_B) \geq \frac{KM}{L} + \begin{cases} 0, & M \equiv 0 \,(\text{mod } 4) \\ \frac{m(m-1)}{L^2}, & M \equiv 1 \,(\text{mod } 2) \\ \frac{4}{L^2}[\lfloor \frac{m}{2} \rfloor^2 + \lceil \frac{m}{2} \rceil^2 - m], & M \equiv 2 \,(\text{mod } 4) \end{cases} \tag{2}$$

where K is the number of signatures, L is the signature length, $M = \max\{K, L\}$, and $m = \min\{K, L\}$. The subscript "B" in $\mathcal{S}_B$ identifies a binary signature set $\mathcal{S}_B \in \frac{1}{\sqrt{L}} \{\pm 1\}^{L \times K}$, and $\lfloor x \rfloor$, $\lceil x \rceil$ stand for the closest to x integer that is less than or equal to x and greater than or equal to x, respectively.

In this paper, we consider quaternary signature sets $\mathcal{S}_Q \triangleq [\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K] \in \frac{1}{\sqrt{L}} \{\pm 1, \pm j\}^{L \times K}$, $j \triangleq \sqrt{-1}$, where the subscript "Q" in $\mathcal{S}_Q$ stands for quaternary. Since binary signature sets are special cases of quaternary signature sets ($\{\pm 1\} \subset \{\pm 1, \pm j\}$), for any K and L any achievable lower bound on the TSC of $\mathcal{S}_Q$ lies between the Welch bound and the Karystinos-Pados bound. Thus, using (2), whenever $M \equiv 0 \,(\text{mod } 4)$ the bound on TSC of any set $\mathcal{S}_Q$ is

$$\text{TSC}(\mathcal{S}_Q) \geq \begin{cases} K, & K \leq L \text{ and } L \equiv 0 \,(\text{mod } 4), \\ K^2/L, & K > L \text{ and } K \equiv 0 \,(\text{mod } 4). \end{cases} \tag{3}$$

In the rest of this section, we derive new bounds on the TSC of quaternary signature sets for all possible combinaitons of the values of K and L when $M \equiv 1 (\mathrm{mod}\ 2)$ and $M \equiv 2 (\mathrm{mod}\ 4)$. Then, via quaternary Hadamard matrix transformations, we design optimal quaternary signature sets that achieve the new bounds.

A. Underloaded system ($K \leq L$)

Theorem 1 below provides new lower bounds on the TSC of any signature set $\mathcal{S}_Q$ when $K \leq L$ (underloaded systems) and $\max\{K, L\} = L$ is not a multiple of 4.

Theorem 1: Let $\mathcal{S}_Q$ be an arbitrary quaternary signature set $\mathcal{S}_Q \triangleq [\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K] \in \frac{1}{\sqrt{L}}\{\pm 1, \pm j\}^{L \times K}$, $j = \sqrt{-1}$, $K \leq L$. Then,

$$\mathrm{TSC}(\mathcal{S}_Q) \geq \begin{cases} K + \frac{K(K-1)}{L^2}, & L \equiv 1(\mathrm{mod}\ 2) \\ K, & L \equiv 0(\mathrm{mod}\ 2). \end{cases} \quad (4)$$

Proof: The TSC of $\mathcal{S}_Q$ can be expressed as

$$\mathrm{TSC}(\mathcal{S}_Q) = K + \sum_{m=1}^{K} \sum_{\substack{n=1, \\ m \neq n}}^{K} \left| \mathbf{s}_m^H \mathbf{s}_n \right|^2 \quad (5)$$

where the double-summation term is the TSC between *different* signatures in $\mathcal{S}_Q$ (if this term has zero value, i.e. all pairs of signatures have zero cross-correlation, the lower bound of $\mathrm{TSC}(\mathcal{S}_Q)$ reduces to the Welch bound). To obtain a lower bound on the double-summation term in (5), we consider the set $\mathcal{C}$ of all non-ordered pairs of signatures $\{\mathbf{s}_m, \mathbf{s}_n\}$, $m \neq n$, with non-zero cross correlation, i.e. $\mathcal{C}\left(\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K\}\right) \triangleq \{\{\mathbf{s}_m, \mathbf{s}_n\}$ such that $m \neq n$ and $\mathbf{s}_m^H \mathbf{s}_n \neq 0$, $m = 1, 2, \ldots, K$, $n = 1, 2, \ldots, K\}$. Then,

$$\mathrm{TSC}(\mathcal{S}_Q) \geq K + 2 \left| \mathcal{C}(\mathcal{S}_Q) \right| |A|^2 \quad (6)$$

where A is a lower bound on the cross-correlation of any two signatures in $\mathcal{S}_Q$ (i.e., $|\mathbf{s}_m^H \mathbf{s}_n| \geq A \ \forall \ \mathbf{s}_m, \mathbf{s}_n \in \mathcal{S}_Q$) and $|\mathcal{C}(\cdot)|$ denotes the cardinality of the set $\mathcal{C}(\cdot)$. Since the quaternary signature alphabet $\frac{1}{\sqrt{L}}\{\pm 1, \pm j\}$ is closed under multiplication and conjugation (denoted by "*"), signature cross-correlations can be expressed as

$$\mathbf{s}_m^H \mathbf{s}_n \triangleq \sum_{i=1}^{L} s_{mi}^* s_{ni} = a \left(\frac{+1}{L} \right) + b \left(\frac{-1}{L} \right) + c \left(\frac{+j}{L} \right) + d \left(\frac{-j}{L} \right)$$

for some integers a, b, c, d such that $a + b + c + d = L$. Then, $\left| \mathbf{s}_m^H \mathbf{s}_n \right| = \frac{1}{L}\sqrt{(a-b)^2 + (c-d)^2}$. If $L \equiv 1(\mathrm{mod}\ 2)$, $a + b + c + d \equiv 1(\mathrm{mod}\ 2)$ which implies that $(a-b)^2 + (c-d)^2 \equiv 1(\mathrm{mod}\ 2)$ and since $(a-b)^2 + (c-d)^2 \geqslant 0$, we have $(a-b)^2 + (c-d)^2 \geqslant 1$. Thus, $\left| \mathbf{s}_m^H \mathbf{s}_n \right| \geq \frac{1}{L}$ for any $\mathbf{s}_m, \mathbf{s}_n \in \mathcal{S}_Q$. We conclude that if $L \equiv 1(\mathrm{mod}\ 2)$ then the cross-correlation value between any two signatures in $\mathcal{S}_Q$ is non-zero ($|A| = \frac{1}{L}$), therefore $|\mathcal{C}(\mathcal{S}_Q)| = \binom{K}{2} = K(K-1)/2$. On the other hand, if $L \equiv 0(\mathrm{mod}\ 2)$ there may be signature pairs in $\mathcal{S}_Q$ that exhibit zero cross-correlation. ∎

The new bounds on the TSC of quaternary signature sets for underloaded systems ($K \leq L$) are summarized in Table I. Table

TABLE I
UNDERLOADED QUATERNARY SEQUENCE SETS ($K \leq L$)

Length	Number of Sequences	Lower Bound on TSC
$L \equiv 0(\mathrm{mod}\ 2)$	Any K	K
$L \equiv 1(\mathrm{mod}\ 2)$	Any K	$K + \frac{K(K-1)}{L^2}$

TABLE II
OVERLOADED QUATERNARY SEQUENCE SETS ($K \geq L$)

Number of Sequences	Length	Lower Bound on TSC
$K \equiv 0(\mathrm{mod}\ 2)$	Any L	$\frac{K^2}{L}$
$K \equiv 1(\mathrm{mod}\ 2)$	Any L	$\frac{K^2}{L} + \frac{L-1}{L}$

I can also be viewed as proof that when the signature length is not even, no orthogonal quaternary signature set exists.

B. Overloaded system ($K > L$)

Let $\mathbf{d}_l = [\mathbf{s}_1(l), \mathbf{s}_2(l), \cdots, \mathbf{s}_K(l)]^T \in \frac{1}{\sqrt{L}}\{\pm 1, \pm j\}^K$ denote the transpose of the lth row, $l = 1, 2, \ldots, L$, of the signature matrix $\mathcal{S}_Q$. Due to the "row-column equivalence" property, $\mathrm{TSC}(\mathcal{S}_Q) = \sum_{m=1}^{K} \sum_{n=1}^{K} \left| \mathbf{s}_m^H \mathbf{s}_n \right|^2 = \sum_{l=1}^{L} \sum_{r=1}^{L} \left| \mathbf{d}_l^H \mathbf{d}_r \right|^2$. Therefore, we can proceed with the calculation of $\mathrm{TSC}(\mathcal{S}_Q)$ as follows

$$\begin{aligned} \mathrm{TSC}(S_Q) &= \sum_{l=1}^{L} \left| \mathbf{d}_l^H \mathbf{d}_l \right|^2 + \sum_{l=1}^{L} \sum_{r=1, r \neq l}^{L} \left| \mathbf{d}_l^H \mathbf{d}_r \right|^2 \\ &= \frac{K^2}{L} + \sum_{l=1}^{L} \sum_{r=1, r \neq l}^{L} \left| \mathbf{d}_l^H \mathbf{d}_r \right|^2 \\ &= \frac{K^2}{L} + 2 \left| C(\{\mathbf{d}_1, \mathbf{d}_2, \cdots, \mathbf{d}_L\}) \right| \left| \mathbf{d}_l^H \mathbf{d}_r \right|^2 \end{aligned} \quad (8)$$

By Theorem 1, we obtain that

$$\mathrm{TSC}(\mathcal{S}_Q) \geq \begin{cases} \frac{K^2}{L}, & K \equiv 0(\mathrm{mod}\ 2) \\ \frac{K^2}{L} + \frac{L-1}{L}, & K \equiv 1(\mathrm{mod}\ 2). \end{cases} \quad (9)$$

The new bounds on the TSC of quaternary signature sets for overloaded ($K > L$) systems are summarized in Table II.

We observe that when $M \triangleq \max\{K, L\}$ is $0(\mathrm{mod}\ 4)$ or $1(\mathrm{mod}\ 2)$ (multiple of four or odd) the TSC lower bounds for quaternary signature sets in Tables I and II are identical to the tight binary Karystinos-Pados (KP) bounds [10], while for $M \equiv 2(\mathrm{mod}\ 4)$, the lower bounds of TSC $(\mathcal{S}_Q)$ in Table I and Table II are less than the corresponding KP bounds for binary signature sets. We also recall that $K = L \equiv 1(\mathrm{mod}\ 4)$ is an open standing problem (the only one) [10]-[12] in optimal binary signature set design and KP-bound-equality sets may in fact exist only for values $K = L = 2x(x+1) + 1$, $x = 1, 2, \ldots$ (i.e. $K = L = 5, 13, 25, 41, 61, \ldots$). We conclude that we can benefit in TSC by moving from the binary to the quaternary alphabet if $\max\{K, L\}/2$ is an odd integer or $K = L \neq 2x(x+1) + 1$, $x = 1, 2, \ldots$. Particularly, when $\max\{K, L\}/2$ is odd, the reduction in TSC is approximately $\frac{4m(m-1)}{L^2}$, $m = \min\{K, L\}$. Moreover, for any K and L, half of the minimum-TSC quaternary sets reach the Welch bound

while only a quarter of the minimum-TSC binary sets do. To illustrate the potential reduction in TSC by quaternary designs, in Fig. 1 we consider systems with four different signature lengths $L = 34, 46, 54, 66$ and plot $\text{TSC}(\mathcal{S}) - K$ (i.e. total squared *cross-correlation*) for both binary and quaternary sets as a function of the number of signatures K. To materialize the potential TSC/multiplexing improvements we need designs that meet the corresponding new quaternary bounds in Tables I and II with equality as described in the following section.

Fig. 1.   Total squared *cross-correlation* versus number of signatures K: (a) $L = 34$, (b) $L = 46$, (c) $L = 54$, and (d) $L = 66$.

III. DESIGN OF MINIMUM TSC QUATERNARY SIGNATURE SETS

Our designs of optimal quaternary signature sets are based on transformations of quaternary Hadamard matrices as in [10].
Definition 1: (Quaternary Hadamard matrix)
Let $\mathbf{H}_Q$ be an N-order square matrix over the quaternary alphabet, i.e. $\mathbf{H}_Q \in \{\pm 1, \pm j\}^{N \times N}$, $j \triangleq \sqrt{-1}$, $N > 0$. $\mathbf{H}_Q$ is a quaternary Hadamard matrix if $\mathbf{H}_Q \mathbf{H}_Q^H = N\mathbf{I}_N$ where $\mathbf{I}_N$ is the $N \times N$ identity matrix. ∎

Tables I and II indicate that when $M = \max\{K, L\}$ is even, the lower bound on the TSC of quaternary signature sets is equal to the real/complex Welch bound and can be achieved by any set $\mathcal{S}_Q$ that has orthogonal rows when $K \geq L$ or orthogonal columns when $K \leq L$. If M is not even, our lower bounds on the TSC of quaternary signature sets in (4), (9) are strictly larger than the Welch bound which implies that there is no quaternary matrix $\mathcal{S}_Q$ that has orthogonal rows when $K \geq L$ or orthogonal columns when $K \leq L$. In other words, a necessary condition for a quaternary Hadamard matrix to exist is that its size is even[1]; equivalently, if M is not even, then quaternary Hadamard matrices do not exist.

As is the case for binary Hadamard matrices, there is no universal procedure to generate quaternary Hadamard matrices

---
[1]A necessary condition for a binary Hadamard matrix to exist is that its size is a multiple of four, except for the trivial cases of size one or two.

---

for all even orders. Binary Hadamard matrices can be considered as a special case of quaternary Hadamard matrices. The generation of binary Hadamard matrices of orders that are multiples of four has been well studied. To generate a quaternary Hadamard matrix with order that is a multiple of four, we may multiple by $\pm j$ any column or row of a binary Hadamard matrix of the same order. Unfortunately, it is not easy to generate a quaternary Hadamard matrix of even order N that is not a multiple of four. Below, we propose two alternative methods for this task. First, we suggest a modified version of the analytical procedure of Diţǎ [17]-[18] that generates $(N-2)^2$ polynomial equations. Our modification incorporates constraints that restrict the phase of the matrix elements to be in the set $\{0, \pi/2, \pi, 3\pi/2\}$. Then, the solutions of the constrained system of $(N-2)^2$ polynomial equations can be used as the entries of a quaternary Hadamard matrix. We note that solving such a constrained system of $(N-2)^2$ polynomial equations is, in general, a computationally complex process and no specific algorithm is available in the literature for this task.

On the other hand, exhaustive search to return all quaternary Hadamard matrices of order N has prohibitively high (equal to 4^{N^2}) complexity even for moderate values of N. The second method that we suggest herein has significantly less computational complexity (equal to 4^N) than exhaustive search. Our method is based on the following lemma [19].

Lemma 1: If $\mathbf{A}_1$, $\mathbf{A}_2$ are two circulant matrices such that $\mathbf{A}_1$, $\mathbf{A}_2 \in \{\pm 1, \pm j\}^{\frac{N}{2} \times \frac{N}{2}}$, $\frac{N}{2} \in \mathbb{N}$, and $\mathbf{A}_1$, $\mathbf{A}_2$ satisfy

$$\mathbf{A}_1\mathbf{A}_1^H + \mathbf{A}_2\mathbf{A}_2^H = N\mathbf{I}_{\frac{N}{2}} \qquad (10)$$

where $\mathbf{I}_{\frac{N}{2}}$ is an $\frac{N}{2} \times \frac{N}{2}$ identity matrix, then the construction $\mathbf{H}_Q = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2^H & -\mathbf{A}_1^H \end{bmatrix}$ is an N-order quaternary Hadamard matrix. ∎

As an example, if $\mathbf{A}_1 = \begin{bmatrix} j & 1 & 1 \\ 1 & j & 1 \\ 1 & 1 & j \end{bmatrix}$
and $\mathbf{A}_2 = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$, then $\mathbf{H}_Q =$
$\begin{bmatrix} j & 1 & 1 & -1 & 1 & 1 \\ 1 & j & 1 & 1 & -1 & 1 \\ 1 & 1 & j & 1 & 1 & -1 \\ -1 & 1 & 1 & j & -1 & -1 \\ 1 & -1 & 1 & -1 & j & -1 \\ 1 & 1 & -1 & -1 & -1 & j \end{bmatrix}$ is a quaternary Hadamard
matrix. There are only $4^{\frac{N}{2}}$ distinct $\frac{N}{2} \times \frac{N}{2}$ circulant matrices over the quaternary alphabet and each of them can be identified by its first row only. If two $\frac{N}{2} \times \frac{N}{2}$ circulant matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ that satisfy (10) exist, they can be found by examining all 4^N possible pairs of circulant matrices. Then, a quaternary Haramard matrix with order N can be generated by $\mathbf{A}_1$ and $\mathbf{A}_2$ as given by Lemma 1. Examples of the first rows of $\mathbf{A}_1$ and $\mathbf{A}_2$-type matrices for different values of N are given in Table III. Additional quaternary Hadamard matrices generated by this method can be found in [20].

In the rest of this section we present a sufficient condition under which the new TSC lower bounds of Tables I and II become tight. Then, we outline a design procedure of

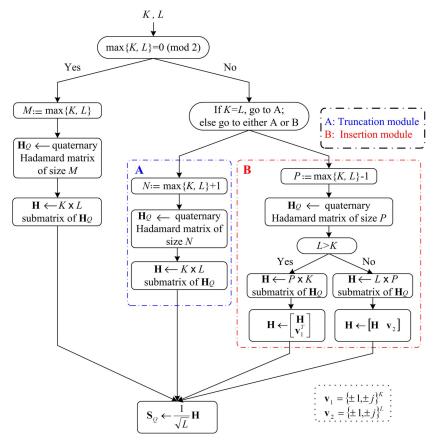|	First row of $\mathbf{A}_1$	First row of $\mathbf{A}_2$
$\frac{N}{2}=3$	$[j\ 1\ 1]$	$[-1\ 1\ 1]$
$\frac{N}{2}=5$	$[1\ j\ -1\ -1\ j]$	$[1\ -1\ j\ j\ -1]$
$\frac{N}{2}=7$	$[1\ 1\ 1\ -1\ -1\ 1\ 1]$	$[1\ j\ -j\ j\ j\ -j\ j]$
$\frac{N}{2}=9$	$[1\ -1\ 1\ 1\ -1\ 1\ -1\ -1\ -1]$	$[1\ 1\ 1\ j\ 1\ 1\ 1\ -1\ -1]$
$\frac{N}{2}=11$	$[1\ -1\ -1\ j\ j\ 1\ 1\ j\ j\ -1\ -1]$	$[j\ 1\ -1\ -j\ j\ -1\ -1\ j\ -j\ -1\ 1]$
$\frac{N}{2}=13$	$[1\ 1\ -1\ j\ j\ -1\ -1\ -1\ -1\ j\ j\ -1\ 1]$	$[j\ 1\ -1\ j\ j\ -1\ -1\ -1\ -1\ j\ j\ -1\ 1]$
$\frac{N}{2}=15$	$[1\ 1\ -1\ j\ -j\ 1\ -1\ -j\ -j\ -1\ 1\ -j\ j\ -1\ 1]$	$[j\ 1\ 1\ j\ j\ -1\ 1\ -j\ -j\ 1\ -1\ j\ j\ 1\ 1]$
$\frac{N}{2}=17$	$[1\ -1\ -1\ j\ -j\ 1\ 1\ j\ j\ j\ j\ 1\ 1\ -j\ j\ -1\ -1]$	$[j\ 1\ -1\ j\ -j\ -1\ -1\ j\ -j\ -j\ j\ -1\ -1\ -j\ j\ -1\ 1]$

Fig. 2.   Optimal quaternary signature set design procedure.

quaternary signature sets that achieve the bounds.

Proposition 1: Set $N \triangleq 2\lceil\max\{K,L\}/2\rceil$ and $P \triangleq 2\lfloor\max\{K,L\}/2\rfloor$. If there exists a quaternary Hadamard matrix of size N, then for any K and L there exists a quaternary signature set $\mathcal{S}_Q = [\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K] \in \frac{1}{\sqrt{L}}\{\pm 1, \pm j\}^{L \times K}$ that achieves the TSC lower bound in Table I or II. If there exists a quaternary Hadamard matrix of size P, then there exists a quaternary signature set $\mathcal{S}_Q = [\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K] \in \frac{1}{\sqrt{L}}\{\pm 1, \pm j\}^{L \times K}$ with $K \neq L$ that achieves the TSC lower bound in Table I or II. ∎

For underloaded systems, $K \leq L$, let $N = 2\lceil L/2\rceil$ and generate an N-order quaternary Hadamard matrix $\mathbf{H}_Q$. Either $L = N$ or $L = N - 1$. If $L = N$, then a quaternary set $\mathcal{S}_Q$ can be formed by selecting and normalizing by $\frac{1}{\sqrt{L}}$ any K columns of $\mathbf{H}_Q$; if $L = N - 1$, then we first truncate $\mathbf{H}_Q$ by one row and then form $\mathcal{S}_Q$ by selecting and normalizing by $\frac{1}{\sqrt{L}}$ any K columns from the truncated matrix. For overloaded systems, $K \geq L$, let $N = 2\lceil K/2\rceil$ and generate an N-order quaternary Hadamard matrix $\mathbf{H}_Q$. Then, $K = N$ or $K = N - 1$. If $K = N$, we may choose any L rows of $\mathbf{H}_Q$ and normalize them by $\frac{1}{\sqrt{L}}$; this is our $\mathcal{S}_Q$. If $K = N-1$, we may proceed by truncating $\mathbf{H}_Q$ by one column and then form $\mathcal{S}_Q$ by choosing and normalizing by $\frac{1}{\sqrt{L}}$ any L rows of the truncated matrix.

By Proposition 1, a minimum-TSC quaternary signature set can also be designed based on a $P = 2\lfloor\max\{K,L\}/2\rfloor$ order quaternary Hadamard matrix if it exists. Since $P = N$ when

$\max\{K, L\} \equiv 0(\bmod\ 2)$, we focus on the case $\max\{K, L\} \equiv 1(\bmod\ 2)$ and $K \neq L$. For underloaded systems, $K < L$, $L \equiv 1(\bmod\ 2)$, and $P = 2\lfloor L/2 \rfloor = L - 1$. Generate an $(L-1)$-order quaternary Hadamard matrix $\mathbf{H}_Q$. To form $\mathcal{S}_Q$, we first select any K columns of $\mathbf{H}_Q$, then insert an arbitrary row vector $\mathbf{v}_1^T \in \{\pm 1, \pm j\}^{1 \times K}$, and finally normalize all columns by $\frac{1}{\sqrt{L}}$. For overloaded systems, $K > L$, $K \equiv 1(\bmod\ 2)$, and $P = 2\lfloor K/2 \rfloor = K - 1$. Generate a $(K-1)$-order quaternary Hadamard matrix $\mathbf{H}_Q$. To form $\mathcal{S}_Q$ we may proceed by choosing any L rows of $\mathbf{H}_Q$, inserting an arbitrary column vector $\mathbf{v}_2 \in \{\pm 1, \pm j\}^{L \times 1}$, and finally normalizing all rows by $\frac{1}{\sqrt{L}}$.

Fig. 2 summarizes the quaternary signature set design procedure described above in the form of a flow chart subject to the existence of a quaternary Hadamard matrix of order $N = 2\lceil \max\{L, K\}/2 \rceil$ or $P = 2\lfloor \max\{L, K\}/2 \rfloor$. We can show that the TSC of sets $\mathcal{S}_Q$ designed by this procedure is exactly equal to the corresponding new bounds in Tables I or II and thus the produced quaternary signature sets are TSC-optimal.

As an illustrative example, in Fig. 3 we give a TSC-optimal quaternary signature set for an overloaded system with signature length $L = 13$ and $K = 22$ signatures. Another example of optimal design with $L = K = 9$ is shown in Fig. 4. These optimal sets were obtained directly by the design procedure of Fig. 2.
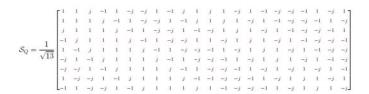

Fig. 3. Optimal quaternary signature set for overloaded multiplexing with signature length $L = 13$ and $K = 22$ signatures.
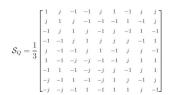

Fig. 4. Optimal quaternary signature set with signature length $L = 9$ and $K = 9$ signatures.

IV. CONCLUSIONS

In this paper, we derived new bounds on the TSC of quaternary signature sets for both underloaded and overloaded code-division multiplexing systems (summarized in Tables I and II, respectively). We showed that the new bounds on the TSC of quaternary signature sets are lower than the corresponding binary signature set bounds (same number of signals K and signature length L) for all $\max\{K, L\} \equiv 2(\bmod\ 4)$ or $K = L \equiv 1(\bmod\ 4) \neq 2x(x+1) + 1$ cases. We then

designed minimum-TSC optimal quaternary sets that meet the new bounds for all K, L. Our design procedure depends on the existence of a quaternary Hadamard matrix of size $2\lceil \max\{K, L\}/2 \rceil$ or $2\lfloor \max\{K, L\}/2 \rfloor$.

The sum-capacity, maximum squared correlation, and total asymptotic efficiency of minimum TSC quaternary sets are evaluated in closed-form and contrasted against minimum-TSC optimal binary and real/complex sets in [21].

REFERENCES

[1] L. R. Welch, "Lower bounds on the maximum cross correlation of signals," *IEEE Trans. Inform. Theory*, vol. 20, pp. 397-399, May 1974.
[2] M. Rupf and J. L. Massey, "Optimum sequence multisets for synchronous code-division multiple-access channels," *IEEE Trans. Inform. Theory*, vol. 40, pp. 1261-1266, July 1994.
[3] D. V. Sarwate, "Meeting the Welch bound with equality," in *Sequences and their Applications: Proceedings of SETA '98*, C. Ding, T. Helleseth, and H. Niederreiter, Eds. London, UK: Springer-Verlag, 1999.
[4] C. Rose, S. Ulukus, and R. D. Yates, "Wireless systems and interference avoidance," *IEEE Trans. Wireless Commun.*, vol. 1, pp. 415-428, July 2002.
[5] P. Viswanath, and V. Anantharam, "Optimal sequences for CDMA under colored noise: A Schur-Saddle function property," *IEEE Trans. Inform. Theory*, vol. 48, pp. 1295-1318, June 2002.
[6] J. A. Tropp, I. S. Dhillon, and R. W. Heath Jr., "Finite-step algorithms for constructing optimal CDMA signature sequences," *IEEE Trans. Inform. Theory*, vol. 50, pp. 2916-2921, Nov. 2004.
[7] O. Popescu and C. Rose, "Sum capacity and TSC bounds in collaborative multibase wireless systems," *IEEE Trans. Inform. Theory*, vol. 50, pp. 2433-2440, Oct. 2004.
[8] G. S. Rajappan and M. L. Honig, "Signature sequence adaptation for DS-CDMA with multipath," *IEEE Journal on Selected Areas in Commun.*, vol. 20, pp. 384-395, Feb. 2002.
[9] P. Xia, S. Zhou, and G. B. Giannakis, "Achieving the Welch bound with difference sets," *IEEE Trans. Inform. Theory*, vol. 51, pp. 1900-1907, May 2005.
[10] G. N. Karystinos and D. A. Pados, "New bounds on the total squared correlation and optimum design of DS-CDMA binary signature sets," *IEEE Trans. Commun.*, vol. 51, pp. 48-51, Jan. 2003.
[11] C. Ding, M. Golin, and T. Kløve, "Meeting the Welch and Karystinos-Pados bounds on DS-CDMA binary signature sets," *Des., Codes Cryptogr.*, vol. 30, pp. 73-84, Aug. 2003.
[12] V. P. Ipatov, "On the Karystinos-Pados bounds and optimal binary DS-CDMA signature ensembles," *IEEE Commun. Lett.*, vol. 8, pp. 81-83, Feb. 2004.
[13] G. N. Karystinos and D. A. Pados, "The Maximum Squared Correlation, Sum Capacity,and Total Asymptotic Efficiency of Minimum Total-Squared-Correlation Binary Signature Sets," *IEEE Trans. Inform. Theory*, vol. 51, pp. 348-355, Jan. 2005.
[14] F. Vanhaverbeke and M. Moeneclaey, "Sum capacity of binary signatures that minimize the total squared correlation," in *Proc. Intern. Symp. Inform. Theory (ISIT)*, Chicago, IL, June 2004, p. 432.
[15] F. Vanhaverbeke and M. Moeneclaey, "Binary signature sets for increased user capacity on the downlink of CDMA systems," *IEEE Trans. Wireless Commun.*, vol. 5, pp. 1795-1804, July 2006.
[16] D. C. Popescu and P. Yaddanapudi, "Narrowband interference avoidance in OFDM-based UWB communication systems," *IEEE Trans. Commun.*, vol. 55, pp. 1667-1673, Sept. 2007.
[17] P. Diţă, "Some results on the parametrization of complex Hadamard matrices," *J. Phys. A: Math. Gen.*, vol. 37, pp. 5355-5374, 2004.
[18] P. Diţă, "Factorization of unitary matrices," *J. Phys. A: Math. Gen.*, vol. 36, pp. 2781-2789, 2003.
[19] D. R. Stinson, *Combinatorial Designs: Constructions and Analysis*, New York: Springer-Verlag, 2004.
[20] Quaternary Hadamard Matrices, [Online]. Available: *http://www.eng.buffalo.edu/˜mingli/Research/Research.html*
[21] M. Li, S. N. Batalama, D. A. Pados, and J. D. Matyjas, "The maximum squared correlation, total asymptotic efficiency, and sum capacity of minimum total-squared-correlation quaternary signature sets," submitted to *EUSIPCO*, Glasgow, Scotland, Aug. 2009.